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# Fractal Structures in Turbulence

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We present a qualitative overview of our work on the issue of fractal structures in turbulence. We explain why fully developed turbulence is not space filling and describe how its fractal dimension can be estimated theoretically. The implications of the fractal nature of turbulence on transport processes like turbulent diffusion and on fluctuations in passive scalars are discussed. The latter affect wave propagation in turbulent media and these effects are examined. In addition we consider clouds in the atmosphere which are claimed to have fractal perimeters (or surfaces) and outline the physical reasons for this phenomenon. The fractal dimension of clouds is tied to the theory of turbulent diffusion and is computed theoretically. Indications of the road ahead are given.

**KEY WORDS:** Fractals; turbulence; passive scalars; clouds; wave propagation.

## 1. INTRODUCTION

Turbulent fluid motions and processes that occur in turbulent fluids form excellent grounds for testing ideas concerning the role of fractals in physics. The turbulent activity (or vorticity) in fully developed turbulence seems to concentrate on a fractal.[1–4] Transport processes are sensitive to the fractal nature of turbulence.[5–7] In addition other fractal objects appear to form in turbulent media. An example is clouds which seem to have fractal surfaces.[8,9] In this paper we present a short review of our work that might serve as an introduction to the issue of fractal structures in turbulence. We begin in Section 2 with some basic ideas on turbulence and explain why a fractal model of turbulence is called for. In Section 3 we discuss why turbulence is a fractal and estimate the fractal dimension theoretically. In Section 4 we turn to transport processes like turbulent diffusion and to the

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fractal dimension of clouds. Fluctuations of passive scalars are discussed in Section 5. Section 6 offers conclusions and some discussion of the road ahead.

## 2. TURBULENCE IN BRIEF

Although in principle one should base the theory of turbulence on the equations of fluid mechanics, much of our intuitive understanding of turbulence stems from the qualitative approach of Kolmogorov of 1941.[10] The essence of this approach is the idea that a fluid is kept turbulent by injecting energy on the macroscopic, or stirring, length scales. On the other hand, energy is lost to heat only on the microscopic length scale where the effect of viscous dissipation becomes important. Denoting the length scales by $l_0$ and $l_d$, respectively, Kolmogorov argued that when the Reynolds number is very high, there exists a wide range of length scales $l$ such that $l_d \ll l \ll l_0$. In this "inertial range" viscous dissipation can be neglected. There exists therefore only one parameter which characterizes the system, and this is the mean energy flux per unit mass per unit time, a flux which persists owing to the cascade of energy from the large length scales to the small ones and which results from nonlinear fluid mechanical processes. Denoting this parameter by $\langle \varepsilon \rangle$, we can use it in dimensional analysis to predict the behavior of various quantities of interest. For example, suppose that we want to estimate the typical velocity difference across a length scale $l$, $v_l$. If this can depend on $\langle \varepsilon \rangle$ only (and of course on $l$ itself), dimensional considerations dictate the scaling law $v_l \sim \langle \varepsilon \rangle^{1/3} l^{1/3}$. This shows for example the well-known fact that the energy is contained in the larger eddies.

The simplicity, usefulness, and beauty of this approach are apparent. Unfortunately, there are flaws. A striking one arises when one measures the correlation function of the fluctuation in the local rate of energy dissipation $\varepsilon(\mathbf{r})$, $\langle \varepsilon(\mathbf{r} + \boldsymbol{l})\, \varepsilon(\mathbf{r}) \rangle$. According to the philosophy discussed above, if $l$ is in the inertial range, one should expect the correlation function to be proportional to $\langle \varepsilon \rangle^2$. Experiments[11-13] show, however, that

$$\langle \varepsilon(\mathbf{r} + \boldsymbol{l})\, \varepsilon(\mathbf{r}) \rangle = \mathrm{const} \langle \varepsilon \rangle^2 (l_0/l)^{\mu} \tag{2.1}$$

where $0.5 \lesssim \mu \lesssim 0.25$. The appearance of $\mu \neq 0$ is related to the phenomenon of intermettency and therefore $\mu$ is referred to as the intermittency exponent. The existence of dimensionless corrections in the form of algebraic fall off in this quantity means of course that dimensionless corrections should be expected in quantities like $v_l$ and in any other quantity of interest as well. Accordingly, dimensional analysis seems to become useless, and the simplicity of the Kolmogorov approach can be lost. Here is where the fractal model comes as a remedy. We shall now argue why fractal turbulence is consistent with results like (2.1). In the next section we discuss the physics that gives rise to fractal turbulence.

Suppose that turbulence is not space filling but rather concentrates on an isotropic fractal of dimension $D$. Suppose further (for simplicity) that the fractal is homogeneous[1] (rather than probabilistic). Then the probability $P(l)$ that a vector $\mathbf{l}$ belongs to the fractal is[14] $P(l) = (l/l_0)^{d-D}$, where $d$ is the Euclidean dimension. Now the dissipation $\varepsilon(\mathbf{r})$ is caused by viscous dampling of motions of size $l_d$. The *average* $\langle \varepsilon(\mathbf{r}) \rangle$ will be written therefore as[14]

$$\langle \varepsilon(r) \rangle \sim (l_d/l_0)^{d-D} \nu \frac{v_d^2}{l_d^2} \tag{2.2}$$

where $\nu$ is the viscosity, $v_d$ is the velocity difference across an active region of size $l_d$, and the factor $(l_d/l_0)^{d-D}$ weights the probability that the volume $l_d^d$ belongs to the active, fractal region. Contributions to the correlation function $\langle \varepsilon(\mathbf{r})\, \varepsilon(\mathbf{r} + \mathbf{l}) \rangle$ can come only from length scales of size $l$ or larger, since only eddies of that size correlate points which are a distance $l$ apart. The probability that both points $\mathbf{r}$ and $\mathbf{r} + \mathbf{l}$ belong to activity of size $l \leqslant l_n \leqslant l_0$ is $\sim(l_n/l_0)^{d-D}$. In adition, when we know that both points $\mathbf{r}$ and $\mathbf{r} + \mathbf{l}$ belong to the active region of size $l_n$, we have to weight the probability that each point separately belongs to activity on size $l_d$. This would give rise to the scaling equation

$$\langle \varepsilon(\mathbf{r})\, \varepsilon(\mathbf{r} + \mathbf{l}) \rangle \sim \sum_{l < l_n < l_0} \left[ \frac{l_n}{l_0} \right]^{d-D} \left[ \left[ \frac{l_d}{l_n} \right]^{d-D} \nu \frac{v_d^2}{l_d^2} \right]^2 \tag{2.3}$$

The largest term in the sum is the one for which $l_n = l$. Taking this term,[14] and using Eq. (2.2), we find

$$\langle \varepsilon(\mathbf{r})\, \varepsilon(\mathbf{r} + \mathbf{l}) \rangle = \langle \varepsilon \rangle^2 (l_0/l)^{d-D} \tag{2.4}$$

Comparing with Eq. (2.4) we see that $\mu = d - D$. This result, which has been obtained originally by Mandelbrot, relates the dimensionless corrections to the codimension and shows that the dimensional analysis can be easily corrected by taking fractal statistics into account. Thus the approach of Kolmogorov should not be discarded, but only modified.[2]

[2] The need to modify the 1941 theory was of course known to Kolmogorov. For a complete discussion of the alternative modification based on Kolmogorov's "log normal" model see Ref. 15.

## 3. WHY FRACTAL?

A central question for the physicist is why is turbulence not space filling. The answer should be found in the dynamics of vortex tubes. Vortex tubes are collections of vorticity lines which at every point $\mathbf{r}$ point in the direction of the vorticity $\boldsymbol{\omega}$, $\boldsymbol{\omega}(\mathbf{r}) \equiv \nabla \times \mathbf{v}(\mathbf{r})$. As is well known,[15-17] the time-dependent strain tensors of a turbulent fluid stretch vortex tubes. As the vortex tubes get stretched the vorticity in the direction of the stretching increases randomly in an approximately exponential manner. As long as viscosity can be neglected, Kelvin's circulation theorem shows that the cross section of the tube must decrease to preserve the circulation: $\omega(0)\, l^2(0) = \omega(t)\, l^2(t)$, where $l$ is the radius of the tube. This is another way of seeing the cascade of energy from large length scales to small ones. Naturally, the process of stretching and shrinking of diameter is accompanied by folding on all length scales due to convective motions. The central point now[4] is that as this process occurs the tubes *cannot intersect themselves*. This property stems directly from Kelvin's circulation theorem as long as viscosity can be neglected as argued in Ref. 4. We thus have a rapidly stretched, randomly folded tube that is "self-avoiding." Clearly such a tube has zero probability to fill space. Thus $\mu = d - D$ may not be zero.

This picture can be pushed further to yield an estimate of the fractal dimension $D$. To this end we want to see how the only length scale which is available, i.e., the tube diameter, appears in the problem. This is hard in $d$ dimensions, but if one cuts the stretched, folded tube with a $(d-1)$-dimensional surface, one sees the $(d-1)$-dimensional "rings" that the tube marks on this surface as it pierces it in and out. Because the "ring" diameter is the only length scale available, it has been argued[4] that the structure of rings should be essentially connected so that there would be no average inter-"rings" distance. It has been also argued[4] that the statistics of this structure is essentially the same as that of lattice animals (or branched polymers) from which one estimates the fractal dimension in $(d-1)$-dimensions to be $2(d+1)/5$ in the Flory approximation. In $d$ dimensions one gets $D = (2d+7)/5$ or

$$\mu = d - D = \frac{3d-7}{5} \tag{3.1}$$

which is in agreement with $\mu = 0.4$ in three dimensions. If one uses Monte Carlo results for lattice animals rather than the Flory approximation one bounds $\mu$ between 0.25 and 0.5 in three dimensions in agreement with experiments.[4] Since $\mu$ must be positive or zero one also interprets Eq. (3.1) to mean that $\mu = 0$ in two dimensions, in agreement with numerical simulations. It has been thus suggested that there may be an interesting relation between turbulence in $d$ dimensions and branched polymer physics in $d-1$ dimensions.

## 4. TURBULENT DIFFUSION AND THE FRACTAL DIMENSION OF CLOUDS

One reason why turbulence is such an important subject is that the atmosphere and the oceans are turbulent media. In these media there occur transport processes[18] (of contaminants, humidity, etc.). In addition, in these media light, radio, and sound waves propagate, and the nature of turbulence affects these processes.[19] Wave propagation is considered in Section 5. Here we discuss turbulent diffusion and its ramifications.

### 1. Diffusion in Fractally Homogeneous Turbulence

Single particle diffusion in turbulent media does not have universal properties, since it is dominated by transport by large eddies which depend on the energy injection mechanism, which is not universal.[20,21] The situation is better for two particles or "relative" turbulent diffusion. Here one is interested in interparticle distances. These are not affected by large eddies which convect pairs of particles together. They are not affected by very small eddies either, since these are poor in energy. Thus interparticle distances are mostly affected by eddies of size comparable to them. If the interparticle distance is within the inertial range, one can expect to find universal behavior.

Consider two particles which are released initially at points $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively.[5,6] Their interparticle distance $\mathbf{R}$, $\mathbf{R} \equiv \mathbf{r}_1 - \mathbf{r}_2$, will change in time owing to the fact that the velocities at position $\mathbf{r}_1$ and $\mathbf{r}_2$ are not the same. Denoting the relative velocity by $\mathbf{V}(t)$ we have

$$\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(\tau)\, d\tau \tag{4.1}$$

In isotropic turbulence $\langle \mathbf{V}(t) \rangle = 0$. Consequently $\langle \mathbf{R}(t) \rangle = \langle \mathbf{R}(0) \rangle$. The variance, however, is changing, leading to the turbulent diffusivity[5,6,20]

$$\frac{d\langle R^2 \rangle}{dt} = 2 \int_0^t \langle \mathbf{V}(t) \cdot \mathbf{V}(\tau) \rangle\, d\tau \tag{4.2}$$

We see that in order to understand turbulent diffusion we have to estimate time correlation functions of velocity differences across a length scale $R$. Such estimates have been attempted in Ref. 5. The essence of the argument has been as follows: The correlation $\langle \mathbf{V}(t) \cdot \mathbf{V}(\tau) \rangle$ is known to be nonstationary. We can assert, however, that there exists a function of scaled time variables $g(x)$ such that

$$\langle \mathbf{V}(t) \cdot \mathbf{V}(\tau) \rangle = \langle \mathbf{V}(t) \cdot \mathbf{V}(t) \rangle g \left( \frac{t-\tau}{t_R} \right) \tag{4.3}$$

where $t_R$ is the typical decay time of velocity differences across a length scale $R$. Substitution in Eq. (4.2) leads to the asymptotic predictions[5]

$$\frac{d\langle R^2 \rangle}{dt} \sim \begin{cases} \langle \mathbf{V}(t) \cdot \mathbf{V}(t) \rangle t, & t \ll t_R \quad (4.4a) \\ \langle \mathbf{V}(t) \cdot \mathbf{V}(t) \rangle t_R, & t \gg t_R \quad (4.4b) \end{cases}$$

As long as one lacks knowledge of $g(x)$ one cannot estimate the turbulent diffusivities at all times. One can, however, estimate $\langle \mathbf{V}(t) \cdot \mathbf{V}(t) \rangle$ and $t_R$. As long as $R$ is in the inertial range, the estimate of $\langle \mathbf{V}(t) \cdot \mathbf{V}(t) \rangle$ is relatively easy. Within the "homogeneous fractal model" of turbulence[1,2,5] one finds

$$\langle \mathbf{V}(t) \cdot \mathbf{V}(t) \rangle \sim \langle \varepsilon \rangle^{2/3} R^{2/3} (R/l_0)^{\mu/3} \tag{4.5}$$

The estimate of $t_R$ is given by the guess

$$t_R \sim \frac{R}{V_R} \tag{4.6}$$

where $V_R$ is the typical velocity difference across a length scale $R$.

The approach presented in Ref. 5 led to the estimates

$$\frac{d\langle R^2 \rangle}{dt} \sim \begin{cases} \langle \varepsilon \rangle^{1/3} R^{4/3} \left( \dfrac{R}{l_0} \right)^{\mu/6} & t \ll t_R \quad (4.7a) \\ \langle \varepsilon \rangle^{1/3} R^{4/3} \left( \dfrac{R}{l_0} \right)^{2\mu/3} & t \gg t_R \quad (4.7b) \end{cases}$$

with $R = \langle R^2 \rangle^{1/2}(t)$.

The results in Eqs. (4.7) agree for $\mu = 0$ with the classical "4/3 law" of Richardson of 1926.[20,11] The corrections due to the fractal nature of turbulence are, however, important. In Eq. (4.7b) we have, with $\mu = 0.4$, 20% correction in the exponent of $R$. Comparison of Eqs. (4.7) with available experiments[5] gave excellent agreement with $\mu$ of the order of 0.4. The results are also in agreement with theoretical estimates by Mori and coworkers.[22]

In order to get the function $g(x)$, and thus obtain the diffusivity at all times, one has to invoke fluid mechanical considerations. Such considerations were discussed in Refs. 6 and 7. The main idea there was to use projection operator techniques in conjunction with the Navier–Stokes equation to derive the form of the correlation function of Eq. (4.3). Within stated approximations one obtains a form that pertains to the inertial as well as to the viscous subranges of length scales, with intermittency effects taken into account. The result was then used to derive differential equations for the variance of interparticle distances and thus for the diffusivity. These equations can be used to study in detail the effects of the fractal nature of turbulence on the diffusion process, as well as to understand the role of molecular diffusivity on the apparent dispersion of contaminants in the atmosphere.[7] The reader is referred to Refs. 6 and 7 for more details.

### 4.2. The Fractal Dimension of Clouds

The fact that clouds in the atmosphere have fractal surfaces has been established in an experiment published by Lovejoy.[8,23] In Ref. 8, Lovejoy describes an investigation of the geometry of satellite- and radar-determined cloud and rain areas which vary over six orders of magnitudes of area sizes. Pictures of clouds and rain areas of sizes from 1 km$^2$ to $1.2 \times 10^6$ km$^2$ were digitized on an approximately rectangular grid. Then, the area and perimeter of the cloud (or rain area) were determined simply by counting the number of picture elements and measuring the length of the cloud boundary. The area–perimeter relation was found to be well represented by the relation

$$P \sim \sqrt{A}^{\bar{D}} \tag{4.8}$$

with $\bar{D} = 1.35 \pm 0.05$. It can be easilly argued[9,23] that this scaling law shows that the fractal dimension $D$ of the perimeter equals $\bar{D}$. In addition, if isotropy is assumed, then the fractal dimension of the surface of clouds is $D_c = 1 + D$, i.e., $2.35 \pm 0.05$ in this case.

The experimental result indicates that clouds and rain areas that span many orders of magnitude of size are self-similar. It also indicates that up to length scales of 1000 km there is no characteristic length scale in the turbulent atmosphere. Above 1000 km one expects that the globe's curvature would start to influence the self-similarity of the dynamical processes.

To tie this observation to theory we have to consider the properties of structure functions. Structure function arise whenever we consider "passive scalars" in the theory of turbulence. A passive scalar is a physical quantity which is affected by the turbulent field but does not affect it. Temperature,

[3] Recent experiments show that the scaling obtained by Lovejoy extends to even smaller length scales. I thank Bob Cahalan for showing me his experimental results prior to publication.

humidity, and contaminants are examples. Owing to the randomness of the velocity field the passive scalar is a random function of space. Denote a passive scalar by $\theta(\mathbf{r})$. The structure function $S(l)$ is defined by

$$S(l) = \langle [\theta(\mathbf{r} + l) - \theta(\mathbf{r})]^2 \rangle \tag{4.9}$$

where again the average is over many repetitions of the experiment. The property of most interest for our purposes is the behavior of $S(l)$ in the limit $l \to 0$. If we find

$$\lim_{l \to 0} S(l) \sim l^{2H} \tag{4.10}$$

with $H < 1$ we then refer to $S(l)$ as a "nondifferentiable volume to line function." It can be shown[9,23] that in $(\theta, x, y, z)$ space the function $\theta(\mathbf{r})$ is characterized by a fractal dimension $D_\theta = 4 - H$. The surfaces defined by $\theta(x, y, z) = \theta_0$ are cuts having fractal dimension $3 - H$. A cloud of quantity $\theta$ would be defined by some fixed value $\theta_r$, and therefore if we could find an appropriate structure function $S(l)$ for the clouds, we would be able to determine the fractal dimensions of clouds as $3 - H$. The fractal dimension of the perimeter will be accordingly $2 - H$.

To see the relevant structure function in the case of clouds, suppose that the function $N(\mathbf{r}, t)$ defines the position of a cloud at time $t$ during one realization of turbulent diffusion; $N(\mathbf{r}, t)$ is unity when the point $\mathbf{r}$ is inside the cloud and zero otherwise. Clearly, a boundary to the cloud should be defined. (In Lovejoy's experiment the "boundary" was chosen according to its temperature). Once a boundary is defined,

$$\int N(\mathbf{r}, t)\, d\mathbf{r} = v \tag{4.11}$$

where $v$ is the volume of the cloud which is delineated by the boundary. We suppose now that the action of the turbulent velocity field is to distort the shape of $N(\mathbf{r}, t)$, but not to change its volume (the latter process occurs only on the smallest scales due to molecular diffusion). Then the probabilities $(1/v)\, p(\mathbf{r}, t)$ that a point $\mathbf{r}$ is in the cloud and the joint probability $(1/v^2)\, p(\mathbf{r}_1, \mathbf{r}_2, t)$ that both points $\mathbf{r}_1$ and $\mathbf{r}_2$ are in the cloud are given by

$$p(\mathbf{r}, t) = \langle N(\mathbf{r}, t) \rangle \tag{4.12}$$

$$p(\mathbf{r}_1, \mathbf{r}_2, t) = \langle N(\mathbf{r}_1, t)\, N(\mathbf{r}_2, t) \rangle \tag{4.13}$$

where we reiterate that the average is on many repetitions of the experiment.

Under the action of the velocity field $N(\mathbf{r}, t)$ twists and folds. If it generates a fractal we expect to find a nondifferentiable volume to line function. This can be conveniently defined by

$$S(\boldsymbol{l}, t) = \frac{1}{v} \int d\mathbf{r} \langle [N(\mathbf{r} + \boldsymbol{l}, t) - N(\mathbf{r}, t)]^2 \rangle \tag{4.14}$$

An elementary calculation shows that

$$S(\boldsymbol{l}, t) = 2[1 - P(\boldsymbol{l}, t)] \tag{4.15}$$

where

$$P(\boldsymbol{l}, t) = \frac{1}{v} \int p(\mathbf{r} + \boldsymbol{l}, \mathbf{r}, t)\, d\mathbf{r} \tag{4.16}$$

From the definitions it is clear that $P(\boldsymbol{l} = 0, t) = 1$ and $\int P(\boldsymbol{l}, t)\, d\boldsymbol{l} = v$. From Eq. (4.15) we see now that $\lim_{l \to 0} S(\boldsymbol{l}, t) = 0$. If we could show that

$$S(\boldsymbol{l}, t) \xrightarrow{l \to 0} f(t)\, l^{2H} \tag{4.17}$$

with $H < 1$ we would be able to calculate the fractal dimension of the cloud, $D_c = 3 - H$. Notice that if $N(\mathbf{r}, t)$ were smooth and regular we would have expected $S(\boldsymbol{l}, t) \sim l^2$ or $H = 1$. This would have led to $D_c = 2$ and $D = 1$ for the perimeters of clouds. Lovejoy's experiment indicates interesting nonanalyticities in the limit $S(l \to 0, t)$.

At this stage turbulent diffusion comes back to the picture.[9] The reason is that $P(\boldsymbol{l}, t)$ is determined by its dynamical equation

$$P(\boldsymbol{l}, t) = \int P(\boldsymbol{l}', 0)\, Q(\boldsymbol{l}, t \mid \boldsymbol{l}', 0)\, d\boldsymbol{l}' \tag{4.18}$$

Here $Q(\boldsymbol{l}, t \mid \boldsymbol{l}', 0)$ is the probability that a pair of diffusing particles that were released initially ($t = 0$) at a vector distance $\boldsymbol{l}'$ apart will find themselves at a vector distance $\boldsymbol{l}$ apart at time $t$. This $Q$ function was already discussed by Richardson[20] (who termed it the "distance-neighbor" function) and was later used extensively by Batchelor.[21] Since the mean square dispersion of a pair of particles is determined uniquely by the $Q$ function

$$\langle l^2 \rangle_{l'}(t) = \int l^2 Q(\boldsymbol{l}, t \mid \boldsymbol{l}', 0)\, d\boldsymbol{l} \tag{4.19}$$

the understanding of the small-$l$ limit of $P(\boldsymbol{l}, t)$ [and therefore $S(\boldsymbol{l}, t)$] depends on understanding turbulent diffusion in the small $l$ limit.

We have approached the problem by seeking a scaling function form for $Q$.[9] This is facilitated by realizing that $Q(\mathbf{l}, t \mid \mathbf{l}', 0)$ goes to $T(\mathbf{l}, t)$ in the limit $\mathbf{l}' \to 0$, where $T(\mathbf{l}, t)$ is the probability of finding a pair of particles (that were released from a point source) at a vector distance $\mathbf{l}$ apart at time $t$. This quantity is very likely to obey a differential equation.

Richardson in 1926 suggested that $T(\mathbf{l}, t)$ obeys the differential equation

$$\frac{\partial T(\mathbf{l}, t)}{\partial t} = \frac{\partial}{\partial \mathbf{l}} [K(l)] \frac{\partial T(\mathbf{l}, t)}{\partial \mathbf{l}} \tag{4.20}$$

with $K(l) = \alpha l^{4/3}$ to account for his "4/3" law for turbulent diffusion. He also in a later paper[24] argued that $K(l)$ should be *independent of time* in order that the diffusion should not differ from day to day. As pointed out by Batchelor[21] this argument overlooks the fact that there is an effective origin of time $t$ defining the commencement of the diffusion. However, Batchelor argued further that $K$ could *only be a function of time* as $K(t)$ is the diffusivity and therefore a statistical average over many repetitions of the experiment. Thus Batchelor suggested

$$\frac{\partial T(\mathbf{l}, t)}{\partial t} = K(t) \frac{\partial}{\partial \mathbf{l}} \cdot \frac{\partial}{\partial \mathbf{l}} T(\mathbf{l}, t) \tag{4.21}$$

where $K(t) = \beta[\langle l^2 \rangle(t)]^{2/3}$ and $\langle l^2 \rangle(t) = (\frac{2}{3}\beta t)^3$. We showed,[9] however, that both these equations lead to incorrect predictions for the fractal dimension of clouds. Richardson's equation grows clouds whose perimeter has the fractal dimension 5/3, whereas Batchelor's equation grows spherical clouds with $D = 1$. Accordingly we suggested the more general equation

$$\frac{\partial T(\mathbf{l}, t)}{\partial t} = \frac{\partial}{\partial \mathbf{l}} K(l, t) \frac{\partial T}{\partial \mathbf{l}} (\mathbf{l}, t) \tag{4.22}$$

and allowed $K(l, t)$ to depend on both $l$ and $t$:

$$K(l, t) = Dt^a l^b \tag{4.23}$$

We note that the exponents $a$ and $b$ *cannot be fixed on dimensional grounds alone.* In fact both Eqs. (4.20) (which is consistent with $a = 0$) and (4.21) (which assumes $a = 2$) are dimensionally correct. Other choices of $a$ and $b$ could be made. Our strategy was therefore to determine $a$ and $b$ by demanding consistency with our *independent* knowledge of the turbulent diffusivity as explained in Section 4.1. When done, we found the scaling form of the $Q$ function, substituted it in Eq. (4.18) and found the small $l$ form of $P(l, t)$. When this was used in Eq. (4.15) we found

$$H = (4 - \mu)/6 \tag{4.24}$$

and accordingly the fractal dimension of the perimeter of clouds is expected to be

$$D = 2 - H = \frac{4}{3} + \frac{\mu}{6} \tag{4.25}$$

Picking the value of $\mu$ according to $0.25 < \mu < 0.5$, we see that

$$1.37 < D < 1.41 \tag{4.26}$$

In excellent agreement with the experiment reported in Ref. 8.

The observation of the fractal dimension of clouds is a good example to the stimulus for theoretical research that can be obtained from experimental discoveries of fractal objects.

## 5. PASSIVE SCALAR FLUCTUATIONS AND WAVE PROPAGATION

There exists a variety of wave-propagation phenomena in turbulent media. All these processes are sensitive to the nature of the fluctuations of passive scalars, in particular of the refractive index and of the temperature. Most experiments are analyzed in terms of the structure functions of passive scalars and their Fourier transforms. Denoting the value of a passive scalar $\theta$ at a point $\mathbf{r}$ by $\theta(\mathbf{r})$, the structure function is written as in Eq. (4.9). The corresponding spectral density $\phi_\theta(\kappa)$ is defined by

$$S(l) = 8\pi \int_0^\infty \left[1 - \frac{\sin \kappa l}{\kappa l}\right] \phi_\theta(\kappa)\, \kappa^2 \, d\kappa \tag{5.1}$$

Thus the analysis of experiments calls for a knowledge of the scaling behavior of the structure functions. When the fractal nature of turbulence is not taken into account, one finds the classical result[11,19] for the structure function $S(l) \sim l^{2/3}$ [the "2/3 (−power) law"]. We shall see now that this result is changed appreciably when the effect of intermittency is included.

To find the scaling behavior in fully developed, fractally homogeneous turbulence, we want to consider fluctuations of linear size $l$, $\theta_l$. The main idea[14,19] is that inhomogeneities of size $l$ appear as the result of fluctuations in the fluid velocity across *active* regions of size $l$, $v_l$. There are no inhomogeneities created in the inactive regions of the turbulent medium. With this in mind we can write down immediately that the rate of creation of $\langle\theta_l^2\rangle$ must scale according to

$$\left[\frac{d}{dt}\langle\theta^2\rangle\right]_l \sim \left[\frac{l}{l_0}\right]^\mu \frac{\theta_l^2 v_l}{l} \tag{5.2}$$

where again $(l/l_0)^\mu$ weights the probability to belong to an active region.

In the inertial range these inhomogeneities then break up to smaller scale structures owing to the turbulent cascade to small length scales. This subdivision continues until the inhomogeneities disappear because of molecular dissipation on a length scale $l'_d$. The rate of disappearance, $\langle N\rangle$, can be always written

$$\langle N\rangle = D\langle(\nabla\theta)^2\rangle \tag{5.3}$$

where $D$ is the appropriate diffusion constant. The length scale $l'_d$ is found by equating the rate of creation (or transfer) to the rate of dissipation at this length scale,

$$\frac{v_{l'_d}\theta^2_{l'_d}}{l'_d}\left[\frac{l'_d}{l_0}\right]^\mu \sim \frac{D\theta^2_{l'_d}}{l'^2_d}\left[\frac{l'_d}{l_0}\right]^\mu \tag{5.4}$$

or $l'_d \sim D/v_{l'_d}$. The dissipation length scale for the energy cascade is defined similarly by $l_d = \nu/v_{l_d}$. In many applications $D \simeq \nu$ and therefore $l'_d \simeq l_d$. For convenience we shall disregard their difference in the following.

In the steady state situation we can equate the rate of transfer (or creation) on scale $l$ with the dissipation on scale $l_d$. Therefore

$$\frac{v_l}{l}\theta_l^2\left[\frac{l}{l_0}\right]^\mu \simeq \langle N\rangle \tag{5.5}$$

or

$$\theta_l^2 \sim \frac{\langle N\rangle l}{v_l}\left[\frac{l}{l_0}\right]^{-\mu} \tag{5.6}$$

In a similar way we can find an expression for $v_l$ in terms of $\langle\varepsilon\rangle$. Equating the energy input on the length scale $l_0$, $\langle\varepsilon\rangle$, to the energy transfer, on length scale $l$ (which only occurs in the active regions), we find

$$\langle\varepsilon\rangle \sim \left[\frac{l}{l_0}\right]^\mu \frac{v_l^3}{l} \tag{5.7}$$

or

$$v_l \sim \langle \varepsilon \rangle^{1/3} \, l^{1/3} \left[ \frac{l}{l_0} \right]^{-\mu/3} \tag{5.8}$$

Using Eqs. (5.6) and (5.8) we have

$$\theta_l^2 \sim \langle N \rangle \langle \varepsilon \rangle^{-1/3} \, l^{2/3} (l/l_0)^{-2\mu/3} \tag{5.9}$$

Remember that $\theta_l^2$ is the square of the passive scalar fluctuations *in an active region.* Next we wish to consider the structure functions $S_\theta(l)$ defined by Eq. (4.9). As before, we find the scaling behavior $S_\theta(l)$ by weighting $\theta_l^2$ by the probability of finding an active region of size $l$:

$$S_\theta(l) \sim \left[ \frac{l}{l_0} \right]^{\mu} \theta_l^2 \sim \langle N \rangle \langle \varepsilon \rangle^{-1/3} \, l^{2/3} \left[ \frac{l}{l_0} \right]^{\mu/3} \tag{5.10}$$

as a final form to be used below we write

$$S_\theta(l) = C_\theta^2 \, l^{2/3} \left[ \frac{l}{l_0} \right]^{\mu/3} \tag{5.11}$$

We see that the structure function contains a universal correction to the classical "2/3 (−power) law" for passive scalars. It is straightforward now to find the intermittency corrections to the spectral density $\phi_\theta(\kappa)$ defined in Eq. (5.9). Using Eq. (5.11) we immediately see that

$$\phi_\theta(\kappa) = A C_\theta^2 \kappa^{-11/3} (\kappa l_0)^{-\mu/3} \tag{5.12}$$

where $A$ is a dimensionless constant. In Ref. 14 we have examined the effect of intermittency on phenomena like scintillation of light sources, scattering of electromagnetic waves, amplitude, and phase fluctuations of sound waves, etc. In all cases we found that the effect of $\mu$ being nonzero are significant and should be taken into account.

## 6. CONCLUSIONS AND THE ROAD AHEAD

We have presented a short qualitative review of fractal structures in turbulence. We explained why is turbulence a fractal and what are the implications of the fractal nature on a variety of processes that occur in turbulent media. In addition we considered fractal clouds, and tied their existence to the theory of transport processes in the atmosphere.

In all this we have used the simplest fractal model possible, i.e., the model of "homogeneously fractal turbulence." There are experimental indications that this model is too simple. If one considers, for example, higher-order velocity structure functions $\langle[\mathbf{v}(\mathbf{r}+l)-\mathbf{v}(\mathbf{r})]^n\rangle$ one finds[25] systematic deviations of the experimental scaling laws from those predicted from this simple model. The reason for that is that turbulence should probably be considered as a probabilistic fractal and then various higher-order correlation functions appear to scale with their own codimensions.[1,26] Work on this is in progress and will be reported in the future.

In addition, very little has been done on the fractal nature of nonisotropic turbulent flows.[4] Almost all flows of technological importance are not isotropic. Elucidating their small-scale structure is a task whose importance cannot be overemphasized, and fractal ideas are likely to play important role.

Last, but not least, is the area of compressible turbulence and cosmic phenomena. Also here, very little has been done fractalwise and progress is likely to be made.

[4] For some interesting recent ideas see Ref. 27.

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